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2008 J. Phys.: Condens. Matter 20 345220

(http://iopscience.iop.org/0953-8984/20/34/345220)

IOP PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **20** (2008) 345220 (4pp) doi:10.1088/0953-8984/20/34/345220

Stable and bistable SQUID metamaterials

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Received 18 March 2008, in final form 15 July 2008
Published 6 August 2008
Online at stacks.iop.org/JPhysCM/20/345220

Abstract

We investigate the permeability of metamaterials composed of superconducting quantum interference devices for a microwave probe field and derive the frequency spectrum of permeability and physical conditions for negative (real part) permeability, which may be essential to future experimental research on the new kinds of metamaterial. We find that both the resonance frequency and the permeability of the structure can be smoothly tuned over a large range via a DC bias field in the sub-hysteresis case, whereas in the hysteresis case bistability on both permeability and resonance frequency can occur. In addition, negative permeability at extremely low frequency (much lower than the resonance frequency of LC oscillation) is possible.

(Some figures in this article are in colour only in the electronic version)

Metamaterials are a new class of electromagnetic materials which exhibit exotic properties and the response of which to an EM wave is dictated by a microstructure carefully engineered on a scale much less than the wavelength. The most familiar metamaterials are those that have simultaneously negative permittivity and negative permeability and therefore have negative refractive index. They are also called left-handed metamaterials (LHMs), the concept of which was proposed by Veselago [1] and attracted much attention after Pendry [2, 3] suggested that they might be fabricated by spit-ring resonators (SRR), which are for negative permeability, and direct wires, which are for negative permittivity. The most familiar and remarkable application of LHMs is for sub-resolution imaging [4]. In contrast to LHMs, single negative metamaterials (which have negative permeability but positive permittivity or vice versa) have also attracted considerable attention because of their unusual interaction with electromagnetic waves [5]. Metamaterials can even be applied to cloaking devices which are possible for use to render objects located inside invisible to observation from the outside [6].

In contrast to natural materials, which are composed of atoms or other quantum particles, the unit cells of most artificial metamaterials are classical particles. Recently we have proposed a metamaterial whose magnetic resonators, SRRs, are replaced by superconducting quantum interference devices (SQUID) and find that negative permeability can be achieved via quantum transitions between states of electric current [7]. After this, Lazarides *et al* [8] show that negative permeability at some individual frequencies can also be achieved via classical dynamics of electric current. However, the frequency spectrum of permeability has not been obtained and its analysis is limited to the sub-hysteresis case [9][1]. In this paper, we will investigate the SQUID metamaterial being tuned by a DC bias field including both sub-hysteresis and hysteresis cases and present the frequency spectrum of permeability for a microwave probe field.

A scheme for a SQUID metamaterial is shown in figure 1, where the SQUIDs are arrayed. We assume the metamaterial is coupled with a weak microwave field, the flux of which shed on the SQUID ring is much smaller than the flux quantum and the DC bias magnetic field. For the sake of simplicity we assume the structure to be two-dimensional, with the geometry of the SQUID to be a cylindrical ring. The radius of the ring is denoted by a and the period of the array is denoted by d. We assume a microwave probe field is interacting with the composite and its wavelength (λ) satisfies the condition $a \ll d \ll \lambda$, in which case the SQUID array can be considered as an effective medium with permeability μ, while the direct interaction among the SQUIDs can be neglected. In practice, we can use the parameters $a \sim 40\ \mu$m (the typical SQUID size is 10–100 μm), $d \sim 400\ \mu$m and $\lambda \sim 4$ mm.

In [7] we have investigated the negative permeability which is generated by the quantum transition between the energy eigenstates of current in SQUIDs. In this paper, however, we will show that the negative permeability can also been generated by the classical dynamics, which is much simpler than the quantum dynamics and may be much easier to be realized in practice.

The permeability μ can be given (according to [10]) by the relation $B(\omega) = \mu_0 H_x(\omega) + \mu_0 F H'(\omega)$, where $H_x(\omega)$

[1] The technique of Fourier–Bessel series expansion being utilized in [8] is only suitable to sub-hysteresis cases, see [9].

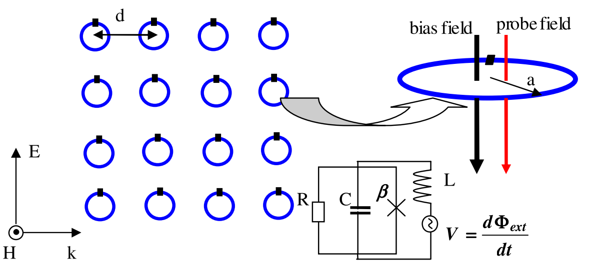

Figure 1. Schematic of a composite metamaterial structure composed of superconducting rings with Josephson junctions (SQUIDs) arrayed and placed in a dielectric background. The permeability for a microwave probe field is tuned by a DC bias field.

is the alternating external magnetic field and $H'(\omega)$ is the additional magnetic field induced by $H_x(\omega)$, which determines the magnetization of the composite, and $F = \pi a^2/d^2$ ($\ll 1$) is the fraction of the structure. Therefore, the permeability can be given by

$$\mu(\omega) = 1 + F\frac{\Phi(\omega) - \Phi_{\mathrm{ext}}(\omega)}{\Phi_{\mathrm{ext}}(\omega)}, \tag{1}$$

where $\Phi_{\mathrm{ext}}(\omega)$ and $\Phi(\omega)$ are the external flux threading and the total flux trapped in the SQUID loop at the microwave field frequency ω, i.e. $\Phi_{\mathrm{ext}}(\omega) = \mu_0 H_x(\omega)\pi a^2$, $\Phi(\omega) = \mu_0(H_x(\omega) + H'(\omega))\pi a^2$. The basic equations of the dynamics of SQUIDs are

$$\frac{Q}{C} = \frac{\mathrm{d}}{\mathrm{d}t}\Phi, \tag{2}$$

$$\Phi = \Phi_{\mathrm{ext}} - LI, \tag{3}$$

$$I = I_{\mathrm{c}}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) + \frac{\mathrm{d}}{\mathrm{d}t}Q + \frac{Q}{C}\frac{1}{R}. \tag{4}$$

If we consider the quantum mechanics of the current in the SQUID, Φ, Q and I in the above equations should be replaced by the operators $\hat{\Phi}$, $\hat{Q}$ and $\hat{I}$, respectively, and $\hat{\Phi}$ and $\hat{Q}$ are connected by the commutation relationship $[\hat{Q}, \hat{\Phi}] = \mathrm{i}\hbar$, which leads to the Heisenberg uncertainty principle $\delta\hat{\Phi}\delta\hat{Q} \geqslant \hbar/2$ ($\delta\Phi \equiv \sqrt{\langle(\Phi - \langle\Phi\rangle)^2\rangle}$, $\delta Q \equiv \sqrt{\langle(Q - \langle Q\rangle)^2\rangle}$). We assume all external fields are classical, i.e. Φ_{ext} is a number (not an operator). In [7] we have studied the case where both $\hat{\Phi}$ and $\hat{Q}$ fluctuate and shown that the quantum transitions between the eigenstates of energy can lead to negative permeability. Here we study the classical limit case where the phase fluctuations $\delta\hat{\Phi}$ can be neglected but the charge fluctuation $\delta\hat{Q}$ is pronounced. Before we use the classical equations (2), (3) and (4), we must know how small $\delta\hat{\Phi}$ should be. In order to evaluate $\delta\hat{\Phi}$, we expand $\sin(2\pi\frac{\Phi}{\Phi_0})$ in equation (4) to second order and take its mean value, which yields $\langle\sin(2\pi\frac{\Phi}{\Phi_0})\rangle \approx \sin(2\pi\frac{\langle\Phi\rangle}{\Phi_0}) - \frac{1}{2}\langle(\Phi - \langle\Phi\rangle)^2\rangle(\frac{2\pi}{\Phi_0})^2\sin(\frac{2\pi\langle\Phi\rangle}{\Phi_0})$, where $\langle\Phi - \langle\Phi\rangle\rangle = 0$ has been used and $\langle\cdots\rangle$ denotes calculating the mean value with quantum mechanics. In order to go back to classical mechanics we must require that in the above equation the second term is negligible comparing with the first term, i.e. $\frac{1}{2}\langle(\Phi - \langle\Phi\rangle)^2\rangle(\frac{2\pi}{\Phi_0})^2 \ll 1$, i.e. $\frac{\delta\Phi}{\Phi_0} \ll \frac{1}{\sqrt{2}\pi}$. This is the condition of the classical limit. In this limit, the classical equations (2), (3) and (4) are valid, but the symbols Φ, Q and I in them should be regarded as $\langle\hat{\Phi}\rangle$, $\langle\hat{Q}\rangle$ and $\langle\hat{I}\rangle$, respectively. On the other hand, the quantum fluctuation of the flux ($\delta\Phi$) can be neglected.

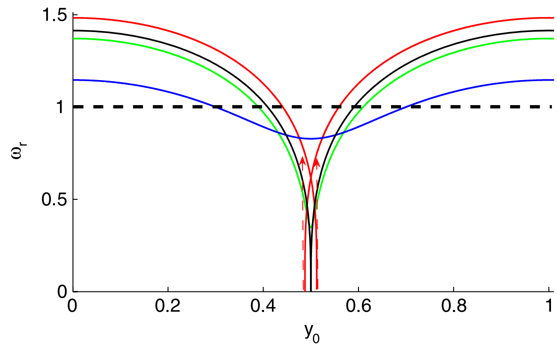

Figure 2. Resonance angular frequency of SQUIDs in units of the resonance angular frequency of LC oscillation versus the external bias DC flux. Black dashed curve for $\beta = 0$, blue solid for $\beta = 0.05$, green solid for $\beta = 0.14$, black solid for $\beta = 1/2\pi \approx 0.1592$ (critical value), red solid for $\beta = 0.1910$.

From equations (2), (3) and (4), it is easy to obtain the equation for time evolution of the flux as follows:

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}\tau} + \beta\sin(2\pi x) + x = y, \tag{5}$$

where x is the normalized total flux defined by $x = \Phi/\Phi_0$, y is the normalized external flux defined by $y = \Phi_{\mathrm{ext}}/\Phi_0$, τ is the normalized time defined by $\tau = \omega_{\mathrm{LC}}t$ with $\omega_{\mathrm{LC}}^2 = \frac{1}{LC}$, γ is the relaxation parameter defined by $\gamma = \omega_{\mathrm{LC}}L/R$ and β is the parameter of a Josephson junction defined by $\beta = \beta_L/2\pi = LI_{\mathrm{c}}/\Phi_0$, where L is the ring inductance, I_{c} is the critical current of the junction, C is the capacitance of the junction and $\Phi_0(= h/2e)$ is the flux quantum. For example, a realistic SQUID system can be described using the parameters as in the work of Zhou *et al* [11], where $L = 100$ pH, $C = 40$ fF and $I_{\mathrm{c}} = 3.95$ μA, leading to $\omega_{\mathrm{LC}} = 5 \times 10^{11}$ rad s^{-1} and $\beta = 1.2/2\pi$.

If the SQUID is in the DC magnetic bias field but the probe microwave field is absent, then $y = y_0 \equiv \Phi_{\mathrm{ext}}^{\mathrm{DC}}/\Phi_0$ with $\Phi_{\mathrm{ext}}^{\mathrm{DC}}$ being the external DC bias flux. From equation (5) we can see that the steady-state solution of the total flux x_0 should obey the following equation:

$$\beta\sin(2\pi x_0) + x_0 = y_0. \tag{6}$$

The 'steady state' will be stable to a infinitely small disturbance only if $\frac{\mathrm{d}y_0}{\mathrm{d}x_0} > 0$, i.e.

$$1 + 2\pi\beta\cos(2\pi x_0) > 0. \tag{7}$$

In contrast, if $\frac{\mathrm{d}y_0}{\mathrm{d}x_0} < 0$, the solution will be unstable. For the bistability case, $\frac{\mathrm{d}y_0}{\mathrm{d}x_0} = 0$ must have solutions, which yield $\beta > 1/2\pi$ (the condition of bistability). In this case when y_0 increases from 0 to ∞, the value of x_0 will jump from the lower branch to the higher branch at a critical point, while when y_0 decreases from ∞ to 0, it will jump from the higher branch to the lower branch at another critical point (see the inset of figure 4).

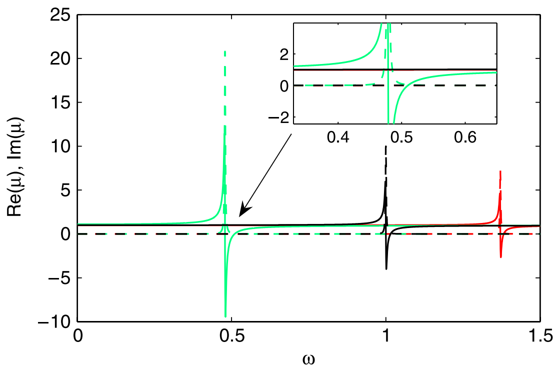

Figure 3. Real part (solid curve) and imaginary part (dashed curve) of the permeability (μ) versus the probe angular frequency ω for the sub-hysteresis case where $F = 0.03$ and $\gamma = 0.003$. The black curve is for $\beta = 0$ and $y_0 = 1.0$, the red curve for $\beta = 0.14$ and $y_0 = 1.0$, and the cyan curve for $\beta = 0.14$ and $y_0 = 0.4874$.

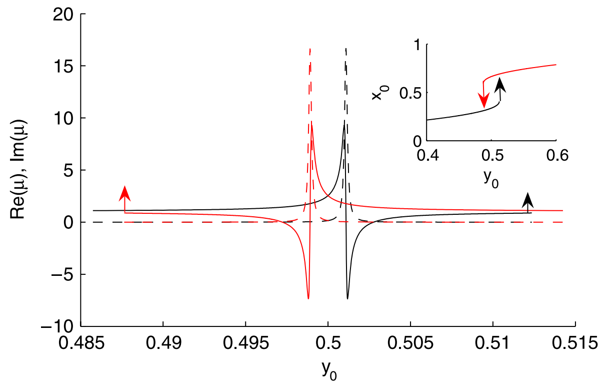

Figure 4. Real part (solid curve) and imaginary part (dashed curve) of the permeability versus y_0 for the hysteresis case where $\beta = 0.1910$. The inset is x_0 versus y_0 which shows the bistable behavior on the input–output relationship of the DC field (without the microwave field). The black curves are for the cases where y_0 varies from small to large, while the red curves for the cases where y_0 varies from large to small. Other parameters are $F = 0.03$, $\gamma = 0.003$ and $\omega = 0.6$ (in units of ω_{LC} ($\omega_{LC} = 1/\sqrt{LC}$)).

Now we consider the case that, after the SQUID reaches a stable steady state, the microwave probe field is switched on. We assume the probe field is linearly polarized with the magnetic field perpendicular to the plane of the SQUID ring. Hence the normalized external magnetic flux y can be written as $y = y_0 + y_1$, where $y_1 = \Phi_{ext}^{probe}/\Phi_0$ with Φ_{ext}^{probe} being the external probe field threading the SQUID loop. Accordingly, the normalized total magnetic flux x can be written as $x = x_0 + x_1$,where $x_0 = \Phi^{DC}/\Phi_0$ and $x_1 = \Phi^{probe}/\Phi_0$, with Φ^{DC} being the total DC flux and Φ^{probe} being the total AC flux threading the SQUID loop. Furthermore, we assume that the probe field is very weak so that $2\pi x_1 \ll 1$. In this case the time evolution of the magnetic flux of the probe field can be described by the following equation:

$$\frac{d^2x_1}{d\tau^2} + \gamma\frac{dx_1}{d\tau} + 2\pi\beta\cos(2\pi x_0)x_1 + x_1 = y_1. \quad (8)$$

For a probe field of angular frequency ω, y_1 can be written as $y_1 = y_a e^{-i\omega\tau}$ +c.c with y_a being the amplitude of the flux of the external field. It is easy to obtain the steady-state solution of equation (8) as $x_1 = x_a e^{-i\omega\tau}$ + c.c with

$$x_a = \frac{-y_a}{\omega^2 + i\omega\gamma - \omega_r^2}, \quad (9)$$

where ω_r is defined by

$$\omega_r = \sqrt{1 + 2\pi\beta\cos(2\pi x_0)}. \quad (10)$$

If the stable condition (i.e. equation (7)) is met, then ω_r must be a real number, and can be considered as the normalized resonance angular frequency of the SQUID (in units of ω_{LC}).

From equations (9) and (10) we can see that ω_r is dependent on x_0. The relation between ω_r and y_0 can be obtained from equations (6) and (10), which is shown in figure 2. If $\beta < 1/2\pi$, ω_r can be continuously and smoothly detuned by y_0. In contrast, if $\beta > 1/2\pi$, ω_r will jump from 0 to a larger value (the bistability on ω_r occurs). It should be noted that, though the case of $\omega_r = 0$ occurs in figure 2, it cannot be obtained in reality because in this case the system is on the brink of instability. However, it can be concluded that in reality $\omega_r \ll 1$ is possible. On the other hand, the maximum of ω_r is $\sqrt{1 + 2\pi\beta}$, which is much larger than 1. So it can be concluded that the resonance frequency of SQUID can be tuned to be much lower or much higher than that of LC oscillations.

The permeability μ can be derived as follows:

$$\mu = 1 - F - \frac{F}{\omega^2 + i\omega\gamma - \omega_r^2}. \quad (11)$$

If we assume γ is small enough so that $\gamma^2 < F$ (e.g. if we take $F = 0.03$, γ should be smaller than 0.1732), it can be derived from the above equation that Re(μ) < 0 occurs only if

$$\gamma^2 < F + 2\omega_r^2 - 2\omega_r\sqrt{\omega_r^2 + F}. \quad (12)$$

This is a necessary condition for negative permeability. The frequency range of Re(μ) < 0 can be obtained as

$$\omega_- < \omega < \omega_+, \quad (13)$$

where

$$\omega_\pm = \frac{\sqrt{F + 2\omega_r^2 - \gamma^2 \pm \sqrt{(F-\gamma^2)^2 - 4\omega_r^2\gamma^2}}}{2}. \quad (14)$$

It can be seen from the above two equations that, when $\omega_r \to 0$, $\omega_- \to 0$. It should be noted that, because $\omega_r = 0$ is impossible in reality, Re(μ) < 0 for zero frequency is impossible. However, negative permeability for extremely low frequency (much lower than the LC oscillation frequency) is possible.

In the sub-hysteresis case the angular-frequency spectra of permeability $\mu(\omega)$ has a Lorentz profile (which is shown in figure 3) and can be tuned smoothly by y_0. Negative Re(μ) occurs when ω is close to ω_r. In the hysteresis case, however, both the response of ω_r and μ to y_0 exhibit bistability (which is shown in figure 4). The high sensitivity of the response of the permeability to the bias field has potential applications in ultra-sensitive detectors.

Let us discuss the condition of the linear approximation. For simplicity we have assumed that the EM probe field is weak enough so that the permeability is a well-defined linear permeability. This condition can be easy to satisfy in the off-resonance case because the current induced by the EM field is very small. In the exact resonance case ($\omega = \omega_r$), however, the linear condition $2\pi x_a \ll 1$ requires that the applied field must satisfy the condition $y_a \ll \gamma\omega_r/2\pi$. In practice it may be difficult to satisfy because γ is very small. In other words, in practice, the response of the metamaterial to a sharp resonant probe field may be nonlinear. However, fortunately, the negative permeability $\mathrm{Re}(\mu) < 0$ does not require sharp resonance. From equations (1) and (9) we can estimate that $\mathrm{Re}(\mu) < 0$ requires $y_a \approx Fx_a$. So the linear condition $2\pi x_a \ll 1$ can be rewritten as $y_a \ll F/(2\pi)$. For example, if we take $F = 0.03$, then the linear condition is $y_a \ll 0.005$. So we can take $y_a = 0.0005 \sim 0.001$ (in units of the flux quantum Φ_0), which is possible in practice.

The above-mentioned discussions are for the case where the probe field is in the weak-field limit ($2\pi x_1 \ll 1$). As a complement, let us have a brief discussion on the case of strong-field limit where $x_1 \gg \beta$. In this case, according to equation (5), the contribution of the Josephson junction can be neglected. Hence the permeability can be given by

$$\mu = 1 - F - \frac{F}{\omega^2 + \mathrm{i}\omega\gamma - 1}. \qquad (15)$$

We emphasize that the permeability in this case is also linear, i.e. independent of y_0, even if $2\pi\beta > 1$. But it is different from the weak-field case because here the permeability is insensitive to the DC bias field and, similar to linear SRR, the SQUID has a constant resonance frequency ($\omega_r = 1$). Hence the tunable properties of the SQUID metamaterial disappear for a strong probe field. It is easy to obtain from equation (15) that the necessary condition for $\mathrm{Re}(\mu) < 0$ is $\gamma < F/\sqrt{2}$. In the case of $\gamma \ll F \ll 1$, the angular frequency range for $\mathrm{Re}(\mu) < 0$ is $1 < \omega < \frac{1}{\sqrt{1-F}}$.

It should be noted that, similar to linear SRR metamaterials, the permeability discussed in this paper is a well-defined linear complex permeability which is independent of the amplitude of the microwave field. This definition of permeability is different from that in [8], where its physical meaning is much more complicated and nonexplicit due to the nonlinearity, and does not seem appropriate for the case of being close to resonance.

In conclusion, we have obtained the spectra of the linear permeability of SQUID metamaterials and present a necessary condition for negative permeability which is given by equation (12). We find negative permeability occurs near the resonance angular frequency ω_r which can be tuned over a large range ($0 < \omega_r < \sqrt{1 + 2\pi\beta}$) via a DC bias magnetic field. Negative permeability at extremely low frequency (much lower than the resonance frequency of LC oscillation) is possible. In the case of hysteresis ($\beta > 1/2\pi$), the response of the resonance frequency and permeability to the DC bias field is hysteretic and bistable. These conclusions may be essential for future experimental research on SQUID metamaterials. The bistable SQUID metamaterials may have potential applications in extremely sensitive detectors due to their high sensitivity to magnetic fields. In addition, the SQUID metamaterials have also potential applications in cloaking devices due to the tunable permeability, especially for cloaking devices at extremely low frequency.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (grant no. 10504016) and the Tsinghua Basic Research Foundation (grant no. JCqn2005037).

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